Fractional statistics in some exactly solvable Calogero-like models with PT invariant interactions

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Abstract

Here we review a method for constructing exact eigenvalues and eigenfunctions of a many-particle quantum system, which is obtained by adding some nonhermitian but PT invariant (i.e., combined parity and time reversal invariant) interaction to the Calogero model. It is shown that such extended Calogero model leads to a real spectrum obeying generalised exclusion statistics. It is also found that the corresponding exchange statistics parameter differs from the exclusion statistics parameter and exhibits a 'reflection symmetry' provided the strength of the PT invariant interaction exceeds a critical value.

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1 Introduction

It is well known that integrable dynamical models and spin chains with long range interactions exhibit fractional statistics or generalised exclusion statistics (GES) [1], which is believed to play an important role in many strongly correlated systems of condensed matter physics. The A_{N-1} Calogero model (related to A_{N-1} Lie algebra) is the simplest example of such dynamical model, containing N particles on a line and with Hamiltonian given by [2, 3]

$$H = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{\omega^2}{2}\sum_{j=1}^{N} x_j^2 + \frac{g}{2}\sum_{j\neq k}\frac{1}{(x_j - x_k)^2}\,, \tag{1.1}$$

where g is the coupling constant associated with long-range interaction. One can exactly solve this Calogero model and find out the complete set of energy eigenvalues as

$$E_{n_1,n_2,\cdots,n_N} = \frac{N\omega}{2}\left[1 + (N-1)\nu\right] + \omega\sum_{j=1}^{N} n_j. \tag{1.2}$$

Here n_js are non-negative integer valued quantum numbers with $n_j \leq n_{j+1}$ and ν is a real positive parameter which is related to g as

$$g = \nu^2 - \nu\,. \tag{1.3}$$

It may be noted that, apart from a constant shift for all energy levels, the spectrum (1.2) coincides with that of N number of free bosonic oscillators. Furthermore, one can easily remove the above mentioned constant shift for all energy levels and express (1.2) exactly in the form of energy eigenvalues for free oscillators: $E_{n_1,n_2,\cdots,n_N} = \frac{N\omega}{2} + \omega\sum_{j=1}^{N}\bar{n}_j$, where $\bar{n}_j = n_j + \nu(j-1)$ are quasi-excitation numbers. However it is evident that these $\bar{n}_j$s are no longer integers and they satisfy a modified selection rule given by $\bar{n}_{j+1} - \bar{n}_j \geq \nu$, which restricts the difference between the quasi-excitation numbers to be at least ν apart. As a consequence, the Calogero model (1.1) provides a microscopic realisation for fractional statistics with ν representing the corresponding GES parameter [4, 5, 6, 7].

Recently, theoretical investigations on different nonhermitian Hamiltonians have received a major boost because many such systems, whenever they are invariant under combined parity and time reversal (PT) symmetry, lead to real energy eigenvalues [8, 9, 10, 11]. This seems to suggest that the condition of hermiticity on a Hamiltonian can be replaced by the weaker condition of PT symmetry to ensure that the corresponding eigenvalues would be real ones. However, till now this is merely a conjecture supported by several examples. Moreover, in almost all of these examples, the Hamiltonians of only one particle in one space dimension have been considered. Therefore, it should be interesting to test this conjecture for the cases of nonhermitian N-particle Hamiltonians in one dimension which remain invariant under the PT transformation [12]:

$$i \to -i, \quad x_j \to -x_j, \quad p_j \to p_j\,, \tag{1.4}$$

where $j \in [1, 2, \cdots, N]$, and x_j ($p_j \equiv -i\frac{\partial}{\partial x_j}$) denotes the coordinate (momentum) operator of the j-th particle. In particular, one may construct an extension of Calogero model by adding to it some nonhermitian but PT invariant interaction, and enquire whether such extended model would lead to real spectrum.

The purpose of the present article is to review the progress [7, 12] on the above mentioned problem for some special cases, where the PT invariant extension of Calogero model can be solved exactly. In Sec.2 of this article we consider such a PT invariant extension of A_{N-1} Calogero model and show that, within a certain range of the related parameters, this extended Calogero model yields real eigenvalues. Next, in Sec.3, we explore the connection of these real eigenvalues with fractional statistics. Section 4 is the concluding section.

2 Exact solution of an extended Calogero model

Let us consider a Hamiltonian of the form [7]

$$\mathcal{H} = H + \delta \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j}, \tag{2.1}$$

where H is given by eqn.(1.1) and δ is a real parameter. It may be observed that though the Hamiltonian (2.1) violates hermiticity property due to the presence of momentum dependent term like $\delta \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j}$, it remains invariant under the combined PT transformation (1.4). Next we recall that, A_{N-1} and B_N Calogero models as well as their distinguishable variants have been solved recently by mapping them to a system of free oscillators [13, 14, 15, 16]. With the aim of solving the PT invariant extension (2.1) of A_{N-1} Calogero model through a similar produce, we assume that (justification for this assumption will be given later) the ground state wave function for this extended model is given by

$$\psi_{gr} = e^{-\frac{\omega}{2} \sum_{j=1}^{N} x_j^2} \prod_{j<k} (x_j - x_k)^{\nu}, \tag{2.2}$$

where ν is a real positive number which is related to the coupling constants g and δ as

$$g = \nu^2 - \nu(1 + 2\delta)\,. \tag{2.3}$$

Now if we use the expression (2.2) for a similarity transformation to the Hamiltonian (2.1), it reduces to an 'effective Hamiltonian' of the form

$$\mathcal{H}' = \psi_{gr}^{-1} \mathcal{H} \psi_{gr} = S^- + \omega S^3 + E_{gr}\,, \tag{2.4}$$

where the Lassalle operator (S^-) and Euler operator (S^3) are given by

$$S^- = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - (\nu - \delta) \sum_{j \neq k} \frac{1}{x_j - x_k} \frac{\partial}{\partial x_j}, \quad S^3 = \sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j}, \tag{2.5}$$

and

$$E_{gr} = \frac{N\omega}{2}\left[1 + (N-1)(\nu - \delta)\right]. \tag{2.6}$$

It is easy to see that the Lassalle operator and Euler operator, as defined in eqn.(2.5), satisfy the simple commutation relation: $[S^3, S^-] = -2S^-$. Using therefore the well known Baker-Hausdorff transformation we can remove the S^- part of the effective Hamiltonian $\mathcal{H}'$ and through some additional similarity transformations reduce it finally to the free oscillator model [7]

$$H_{free} = \mathcal{S}^{-1}\left(\mathcal{H}' - E_{gr}\right)\mathcal{S} = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{\omega^2}{2}\sum_{j=1}^{N} x_j^2 - \frac{\omega N}{2}, \tag{2.7}$$

where $\mathcal{S} = e^{\frac{1}{2\omega}S^-} e^{\frac{1}{4\omega}\nabla^2} e^{\frac{\omega}{2}\sum_{j=1}^{N} x_j^2}$ and $\nabla^2 = \sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}$.

Due to similarity transformations in (2.4) and (2.7), one may naively think that the eigenfunctions of the extended Calogero model (2.1) can be obtained from those of free oscillators as: $\psi_{n_1,n_2,\cdots,n_N} = \psi_{gr}\mathcal{S}\left\{\prod_{j=1}^{N} e^{-\frac{\omega}{2}x_j^2} H_{n_j}(x_j)\right\}$, where n_js are arbitrary non-negative integers and $H_{n_j}(x_j)$ denotes the Hermite polynomial of order n_j. However it is easy to check that, similar to the case of A_{N-1} Calogero model [13], the action of $\mathcal{S}$ on free oscillator eigenfunctions leads to a singularity unless they are symmetrised with respect to all coordinates. Therefore, nonsingular eigenfunctions of the extended Calogero model (2.1) can be obtained from the eigenfunctions of free oscillators as

$$\psi_{n_1,n_2,\cdots,n_N} = \psi_{gr}\ \mathcal{S}\Lambda_+\left\{\prod_{j=1}^{N} e^{-\frac{\omega}{2}x_j^2} H_{n_j}(x_j)\right\}, \tag{2.8}$$

where Λ_+ completely symmetrises all coordinates and thus projects the distinguishable many-particle wave functions to the bosonic part of the Hilbert space. Evidently, the eigenfunctions (2.8) will be mutually independent if the excitation numbers n_js obey the bosonic selection rule: $n_{j+1} \geq n_j$. Thus, in spite of the fact that the interacting Hamiltonian (2.1) is convertible to the free oscillator model, the need for symmetrization shows that the many-particle correlation is in fact inherent in this model. The eigenvalues of the Hamiltonian (2.1) corresponding to the states (2.8) will naturally be given by [7]

$$E_{n_1,n_2,\cdots,n_N} = E_{gr} + \omega\sum_{j=1}^{N} n_j = \frac{N\omega}{2}\left[1 + (N-1)(\nu-\delta)\right] + \omega\sum_{j=1}^{N} n_j. \tag{2.9}$$

Since δ and ν are real parameters, the energy eigenvalues (2.9) are also real ones. Thus we interesting find that the nonhermitian PT invariant Hamiltonian (2.1) yields a real spectrum. Furthermore, it is evident that for all $n_j = 0$, the energy $E_{n_1,n_2,\cdots,n_N}$ attains its minimum value E_{gr}. At the same time, as can be easily worked out from eqn.(2.8), the corresponding eigenfunction reduces to ψ_{gr} (2.2). This proves that ψ_{gr} is indeed the ground state wave function for Hamiltonian (2.1) with eigenvalue E_{gr}.

It may be observed that the eigenfunctions (2.8) pick up a phase factor $(-1)^\nu$ under the exchange of any two particles. Therefore, ν represents the exchange statistics parameter for the extended Calogero model (2.1). By solving the quadratic eqn.(2.3), one can explicitly write down ν as a function of g and δ as

$$\nu = (\delta + \frac{1}{2}) \pm \sqrt{g + (\delta + \frac{1}{2})^2}. \tag{2.10}$$

For the purpose of obtaining real eigenvalues (2.9) as well as nonsingular eigenfunctions (2.8) at the limit $x_i \to x_j$, we have assumed at the beginning of this section that ν is a real positive parameter. This assumption leads to a restriction on the allowed values of the coupling constants g and δ in the following way. First of all, for the case $g < -(\delta + \frac{1}{2})^2$, eqn.(2.10) yields two imaginary solutions. Secondly, for the case $\delta < -\frac{1}{2},\ 0 > g > -(\delta + \frac{1}{2})^2$ eqn.(2.10) yields two real but negative solutions. Inequalities corresponding to these two cases represent two forbidden regions of (δ, g) plane which are excluded from our analysis.

For the case $g > 0$ with arbitrary value of δ, one gets a real positive and a real negative solution from eqn.(2.10). The real positive solution evidently leads to physically acceptable set of eigenfunctions and corresponding eigenvalues within this allowed region of (δ, g) plane. Finally we consider the parameter range $\delta > -\frac{1}{2},\ \ 0 > g > -(\delta + \frac{1}{2})^2$, for which eqn.(2.10) yields two real positive solutions. It is easy to see that these two real positive solutions are related to each other through a 'reflection symmetry' given by $\nu \to 1 + 2\delta - \nu$. Consequently, for each point on the (δ, g) plane within this allowed parameter range, one obtains two different values of the exchange statistics parameter leading to two distinct sets of physically acceptable eigenfunctions and eigenvalues. Thus we curiously find that a kind of 'phase transition' occurs at the line $\delta = -\frac{1}{2}$ on the (δ, g) plane. For the case $\delta > -\frac{1}{2}$, exchange statistics parameter shows the reflection symmetry when g is chosen within an interval $-(\frac{1}{2} + \delta)^2 < g \leq 0$. On the other hand for the case $\delta \leq -\frac{1}{2}$, such reflection symmetry is lost for any possible value of g.

We have seen in this section that, similar to the case of A_{N-1} Calogero model, the extended model (2.1) can also be solved by mapping it to a system of free harmonic oscillators. So it is natural to enquire whether this extended model is directly related to the A_{N-1} Calogero model through some similarity transformation. Investigating along this line [12], we find that

$$\Gamma^{-1} \mathcal{H} \Gamma = H' = \frac{1}{2} \sum p_j^2 + \frac{1}{2} \omega^2 \sum x_j^2 + g' \sum_{j \neq k}^{N} \frac{1}{(x_j - x_k)^2} , \tag{2.11}$$

where $\Gamma = \prod_{j<k} (x_j - x_k)^\delta$, and H' denotes the Hamiltonian of A_{N-1} Calogero model with 'renormalised' coupling constant given by $g' = g + \delta(1+\delta)$. Due to the existence of such similarity transformation, one may expect that the Hamiltonians $\mathcal{H}$ and H' always lead to exactly same eigenvalues. However it should be noted that, within a parameter range given by $\delta > 0,\ g > -\delta(1+\delta)$, there exists a positive solution of eqn.(2.3) satisfying the

condition $\nu - \delta < 0$. Therefore, we can not get any lower bound for the corresponding energy eigenvalues (2.9) at $N \to \infty$ limit. On the other hand, the energy eigenvalues (1.2) of A_{N-1} Calogero model are certainly bounded from below for all possible choice of N and g. So there exists a parameter range within which the spectrum of extended Calogero model differs qualitatively from the spectrum of the original Calogero model. To explain this rather unexpected result, we first observe that the renormalised coupling constant g' would be a positive quantity within the above mentioned parameter range. Consequently, the corresponding exclusion statistics parameter ν', which is obtained by solving eqn.(1.3), has one positive and one negative solution. One usually throws away this negative solution of ν', since the corresponding eigenfunctions become singular at the limit $x_j \to x_k$. However, by using the relation (2.11), such singular eigenfunctions (denoted by $\psi'(x_1, x_2, \cdots, x_N)$) may now be used to construct the eigenfunctions of extended Calogero model (denoted by $\psi(x_1, x_2, \cdots x_N)$) as

$$\psi(x_1, x_2, \cdots, x_N) \ = \ \prod_{j<k} (x_j - x_k)^\delta \psi'(x_1, x_2, \cdots, x_N) \, . \tag{2.12}$$

It can be easily checked that, due to the existence of the factor $\prod_{j<k}(x_j - x_k)^\delta$, the r.h.s. of the above equation becomes nonsingular at the limit $x_j \to x_k$. Thus we curiously find that singular eigenfunctions of H' can be used to generate nonsingular eigenfunctions of $\mathcal{H}$ through the relation (2.12). This shows that the similarity transformation (2.11) is a subtle one and, within a certain parameter range, eigenvalues of extended Calogero model will match with those of Calogero model (having renormalised coupling constant) only if the corresponding unphysical eigenfunctions are taken into account.

3 Connection with fractional statistics

We have mentioned in Sec.1 that GES can be realised microscopically in A_{N-1} Calogero model with hermitian Hamiltonian. The GES parameter for this Calogero model is a measure of 'level repulsion' of the quantum numbers generalising the Pauli exclusion principle. Now for exploring GES in the case of PT invariant model (2.1), we observe that eqn.(2.9) can be rewritten [7] exactly in the form of energy spectrum for N free oscillators as

$$E_{n_1, n_2 \cdots n_N} = \frac{N\omega}{2} + \omega \sum_{j=1}^{N} \bar{n}_j \, , \tag{3.1}$$

where

$$\bar{n}_j = n_j + (\nu - \delta)(j - 1) \tag{3.2}$$

are quasi-excitation numbers. However, from eqn.(3.2) it is evident that such quasi-excitation numbers are no longer integers and satisfy a modified selection rule: $\bar{n}_{j+1} - \bar{n}_j \geq \nu - \delta$. Since the minimum difference between two consecutive $\bar{n}_j$s is given by

$$\tilde{\nu} = \nu - \delta, \tag{3.3}$$

the spectrum of extended A_{N-1} Calogero model (2.1) satisfies GES with parameter $\tilde{\nu}$ [7]. Several comments about this GES parameter are in order. It may be to noted that for $\delta \neq 0$, the GES parameter $\tilde{\nu}$ is different from the power index ν, which is responsible for the symmetry of the wave function. Therefore we may interestingly conclude that unlike Calogero model, the exclusion statistics for model (2.1) differs from its exchange statistics. Furthermore it is already noticed that, on a region of (δ, g) plane satisfying the inequalities $\delta > 0, \ g > -\delta(1+\delta)$, there exists a positive solution of eqn.(2.3) which yields a negative value of $\tilde{\nu}$. For this case, however, one does not get well defined thermodynamic relations at $N \to \infty$ limit and, therefore, can not interpret $\tilde{\nu}$ as the GES parameter.

By using eqn.(2.3) and (3.3), we find the relation

$$\tilde{\nu}^2 - \tilde{\nu} = g + \delta(\delta+1), \tag{3.4}$$

which clearly describes a parabolic curve in the coupling constant plane (δ, g) for any fixed value of $\tilde{\nu}$. As a consequence of this, the competing effect of the independent coupling constants g and δ can make the GES feature of (2.1) much richer in comparison with the Calogero model. For example, while bosonic (fermionic) excitations in the Calogero model occur only in the absence of long-range interaction, the quasi-excitations in (2.1) can behave as pure bosons (fermions) even in the presence of both the long-range interactions satisfying the constraint $\tilde{\nu}(\delta, g) = 0 \ \ (\tilde{\nu}(\delta, g) = 1)$. Both of these constraints lead to the same parabolic curve $g = -\delta(1+\delta)$. A family of such parabolas with shifted apex points are generated for other values of $\tilde{\nu}$ and the lowest apex point is attained at $\tilde{\nu} = \frac{1}{2}$, where the quasi-excitations would behave as semions.

4 Conclusion

Here we construct a many-particle quantum system (2.1) by adding some nonhermitian but combined parity and time reversal (PT) invariant interaction to the A_{N-1} Calogero model. By using appropriate similarity transformations, we are able to map this extended Calogero model to a set of free harmonic oscillators and solve this model exactly. It turns out that this many-particle system with nonhermitian Hamiltonian yields a real spectrum. This fact supports the conjecture that the condition of hermiticity on a Hamiltonian can be replaced by the weaker condition of PT symmetry to ensure that the corresponding eigenvalues would be real ones. It is also found that the spectrum of extended Calogero model obeys a selection rule which leads to generalised exclusion statistics (GES).

However, this extended Calogero model exhibits some remarkable properties which are absent in the case of usual Calogero model. For example, we curiously find that the GES parameter for this extended Calogero model differs from the corresponding exchange statistics parameter. Moreover a 'reflection symmetry' of the exchange statistics parameter, which is known to exist for A_{N-1} Calogero model, can be found in the case

of extended model only if the strength of PT invariant interaction exceeds a critical value.

Finally we note that, it is possible to obtain another exactly solvable many-particle quantum system by adding some nonhermitian but PT invariant interactions to the B_N Calogero model (associated with B_N Lie algebra) [12]. Such a PT invariant model also leads to real spectrum with properties quite similar to the case of extended A_{N-1} Calogero model.

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